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From Practice to New Concepts: Geometric Properties of Groups

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Résumé : Cet article cherche à montrer comment la pratique mathématique, particulièrement celle admettant des représentations visuelles, peut conduire à de nouveaux résultats mathématiques. L'argumentation est basée sur l'étude du cas d'un domaine des mathématiques relativement récent et prometteur: la théorie géométrique des groupes. L'article discute comment la représentation des groupes par les graphes de Cayley rendit possible la découverte de nouvelles propriétés géométriques de groupes.

Abstract: The paper aims to show how mathematical practice, in particular with visual representations, can lead to new mathematical results. The argument is based on a case study from a relatively recent and promising mathematical subject—geometric group theory. The paper discusses how the representation of groups by Cayley graphs made possible to discover new geometric properties of groups.

Introduction

In practice, it seems that mathematics is not carried on solely in terms of axioms, theorems and proofs. Often, some properties are easy to overlook from the perspective given by a definition or a traditional representation. An effective method of revealing new properties is by using different *representations* of the concepts in such a way that these properties become noticeable.

The example demonstrates how groups were represented as graphs. Then the latter represented as metric spaces helped to reveal many geometric properties of groups. As a result, many combinatorial problems were solved through the application of geometry and topology.

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For the epistemological analysis of this example, I apply the approach proposed by Manders [Manders 1999]. He considers mathematical practice as control of the *selective* response to given information, where selective response means *emphasising* some properties of an object while *neglecting* others. From this perspective, representations serve to implement the principal constituents of this activity. As will be demonstrated below, this approach makes clear that a change in representation is a valuable means of finding new properties.

The structure of the paper is as follows. The first section explains Manders' approach and his comparison of Euclid's and Descartes' geometries. The second section presents the case study, namely, it explains how groups previously studied algebraically were approached as geometric objects. This was realised through the representation of groups by Cayley graphs, and then through representing graphs as metric spaces. Detailed definitions will be provided, but for the purpose of this paper it is enough to grasp the general idea. Finally, the conceptual impact of the geometric approach to groups will be analysed in terms of Manders' approach. These issues are developed in detail in Starikova [Starikova 2011].

1 Mathematical practice in the terms of selective responses

In his unpublished paper 'Euclid or Descartes? Representation and Responsiveness', Manders analyses the contribution of *Géométrie*, compared to Euclid's plane geometry. He particularly stresses the epistemic role of the *algebraic notation* in that contribution. He draws attention to the fact that mathematical problem solving has a *strategically selective* character: at each segment of practice *only some* information is taken into account, but not all. Strategical *indifference* includes such epistemic activities as abstraction, unification, idealisation and approximation. *Responding* to particular elements of the context and remaining *indifferent* to others provides control over each step of the context:

> ...[T]he mathematician faced with a proposition to prove must exercise strategically allocated indifference in responding to that situation, say, details of examples that do not bear on general claims one is trying to frame or prove. [Manders 1999, 4]

Manders uses the term 'respondif' for these responses and indifferences. I will refer to them as 'selective responses' (which include indifferences).

In short, Manders announces that two things are to be demonstrated:

> a. that mathematical advance is based on a systematic, coordinated use of responsiveness and indifference, and
>
> b. that the coordination of this responsiveness and indifference is implemented by means of representations [Manders 1999, 2].

Let me explain the key ideas of his approach.

1.1 Application and applicative response

Selective responses are often applied from another domain. This way one can distinguish between direct and application-mediated responses. Further on, I will use the 'original' and 'applicative' response terms correspondingly. The purpose of application is to access the resources of the applied domain. For example, in Descartes' geometry, geometric problems are solved through solving algebraic equations, which represent the geometric curves. Here is Manders' summary of Descartes' approach: [1]

From a diagram-based problem to initial equations.
- Set out the problem in diagram form, as if it is already solved.
- Elaborate the diagram, consider its lines and what will be the primitives in the algebraic equations (constants and variables).
- List the algebraic conditions for the problem.
- Label the known and unknown line segments with single letters.

Algebraic manipulation.
- Eliminate the auxiliary unknowns.
- Reduce the equation to a 'normal form'.
- Analyse how the degrees of the equations can be reduced.

Concluding geometrical steps.
- 'Construct' (solve geometrically) the equation.
- Select the root appropriate to the original problem from amongst those constructed.
- Make a Euclidean-style demonstration, to show that the root selected solves the problem.

Each stage of this process gives rise either to selective responses to the original context or to the applicative context. According to Manders, what makes Descartes' approach exceptionally effective is that it enriches traditional geometry with many *new* types of selective responses. It also *coordinates* them with each other and the already established Euclidean geometrical responses and indifferences. For example, the Cartesian approach *applies* algebraic selective responses:

> Descartes' geometrical method enhances geometry in the style of the ancients by the algebra of his time, by introducing algebraic responses to geometrical problems (systems of equations), algebraic methods to simplify systems of equations and geometrical responses to equations. [Manders 1999, 8]

1. Manders examines the problem of Heraclides in Pappus' *Collectio* as approached by Descartes in Book III of *Géométrie*: given the square AD and the line BN, extend the side AC to E so that EF, on EB with F on CD, equals BN. See [Manders 1999, 13–14].

1.2 The role of representations (artefacts)

In mathematical reasoning we often *produce* and *respond to* artefacts: natural language expressions, Euclidean diagrams, algebraic or logical formulas. According to Manders, artefacts help to *implement* and *control* selective responses. Artefacts also provide new responses, suitable for the current context: e.g. to recognise a region bounded by three sides. They also make the artefacts available for further steps: e.g. to draw a line between two points. Therefore artefacts *fix* and *stabilise* the responses and indifferences to particular elements of a structure.

There is also a coordination of the diagram and the text: the text specifies what is essential in producing and reading the diagrams. Therefore Euclidean constructions involve a complex coordination and control of selective responses both to the diagram and the text.

Such control and coordination may have different levels of 'quality', and this is the point where improvement is possible. In Euclid's demonstrations some information may be read only from a diagram, some only inferred from prior text entries, while some is diagram-based and text-based. For example, in the demonstration of Euclid I.1, we know that the closed curves are circles only from prior stipulation in the text. The fact that they intersect arises only from their situation in the diagram as in the figure below, rather than being inferred from the prior text.

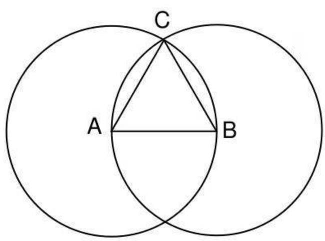

FIGURE 1: Euclid's proposition I.1.

Diagrams are *produced according* to the specifications in the accompanying text, which makes the depicted relations reproducible and therefore stable. Diagram and text contribute to each other differently, and compensate for each others' weaknesses.

However, Manders admits that our control over (Euclidean) diagrams is limited. In the Euclidean style of using diagrams it is possible that some metric properties are unclear. For example, some segments or angles may appear as equal, when in fact they are non-equal. Also there is a risk of missing cases in

Euclidean diagrammatic practice (e.g. the case of an obtuse triangle may be missed with considerable consequences).

Manders specifies that the features that we read directly from a diagram are (a) perceptually explicit, and (b) stable under the diagram perturbations such as a sequence of reproductions of the diagram. These conditions compensate for our limited control of diagrams.

Therefore the Euclidean use of diagrams extensively relies on the *appearance* of the diagrams, which means that the diagrams have to be realised in a medium, and a degree of accuracy—to satisfy conditions (a) and (b)—is required. Furthermore, the non-trivial nuances, such as the solutions' dependence on case branching, must be taken into account.

Let us return to the stages of Descartes' approach. The second stage is paramount for an applicative approach. This stage brings about a substantial advance in problem solving: after the equations are written down, all the calculations proceed quickly by means of efficient algebraic algorithms and become largely a "cut-and-dried matter" [Manders 1999, 28].

Note that the second stage consists mainly of the algebraic manipulations. Indeed, we do not need any more geometrical consideration of known and unknown quantities in the diagram. The coefficients extracted from the diagram at the beginning are later on treated indifferently to their geometrical magnitudes. This is a case of *indifference* to the diagram metric properties and the *application* of an algebraic response—solving equations. After the first stage, our reasoning is neither motivated by, nor particularly responsive to, the diagram of the original problem. As Manders put it:

> Compared to traditional geometrical analysis/synthesis, exclusively compared to "purely synthetic" geometrical thought, the elaborate and clear-cut segmentation of Cartesian geometric problem solving method into distinct tasks and stages, at moderate cost of ultimate geometrical grasp, turns on greatly enhanced respondif coordination and control, especially in the algebraic stages... [Manders 1999, 25]

The final step is just to check if the solutions are relevant. But what initiates the application is the first or representational stage. Manders calls it a "response-shaping" coordination. He makes the important point that the introduction of algebraic notation helps us to apply the fast algebraic algorithms:

> Changing the artefact basis of part of the analytic process, to a representation (algebraic equations) indifferent to diagram appearance, allows all this progress towards solving problems, unimpeded by the difficulties of diagram control in Euclidean-style reasoning. [Manders 1999, 18]

Therefore, in the case of an applicative approach, the representation facilitates the application. Moreover, it endorses a new algebraic notion to geometry—

the degree of an equation. Degree is a new concept unknown in Euclidean practice, which now becomes available for approaching geometrical problems.

Manders remarks that in terms of such criteria as proof-strength, consistency and computability, nothing is significantly improved in Descartes' geometry (except for computability). Also, viewing Descartes' method as simply the elimination of (tedious) geometric constructions would be incomplete. The results of Descartes' algebraisation of geometry are fundamental, and Manders' approach does explain the (conceptual) advance of Descartes' geometry over Greek geometry. It gives us concrete examples of the benefits. One such example is the indifference towards case distinctions, which are not needed in Descartes' algebraic method. Another example is the *new type of response*—the appreciation of the equation degree.

2 The case study

Let me now move to geometric group theory. I shall start with explaining the geometric approach, and then when analysing it I will make the comparison with the combinatorial approach with its focus on the role of representations.

The geometric perspective on groups is not new. Groups were commonly seen as transformations of geometric objects, and this view was the core of Klein's Erlangen programme which aimed to classify and characterise geometries on the basis of group theory (and projective geometry) [Klein 1873].[2] However, in this paper I discuss a new approach, which is based on the idea that groups as such can be thought of as geometric objects. I shall consider hyperbolic groups as an important example and then give it a philosophical analysis.

2.1 The geometric approach

2.1.1 Generated groups

Definition (a generating set). Let G be a group. Then a subset $S \subseteq G$ is called a *generating set* for the group G if every element of G can be expressed as a product of the elements of S or the inverses of the elements of S.

In other words, every element of G can be written as a composition of symbols (called *letters*) representing the elements of S and their inverses. A representation of a non-identity element s as a product of $n \geq 1$ letters is called a *word*. In a given word the number $n \in \mathbb{N}$ is called the *length* of the word. The word with the length equal to 0 is called an *empty word* and by definition it

2. For a historical exposition, see [Hawkins 1984].

represents the group identity I. Other representations of I by words of length $n \geq 1$ are called group *relations*.

There may be several generating sets for the same group. The largest generating set is the set of all group elements. For example, the subsets $\{1\}$ and $\{2, 3\}$ generate the group $(\mathbb{Z}, +)$ or $\mathbb{Z}$ for short, whereas $\{2\}$ does not.

Definition (a finitely generated group). A group with a specified set of generators S is called a *generated group* and is designated as (G, S). If a group has a finite set of generators, it is called a *finitely generated group*.

Example. The group $\mathbb{Z}$ is a finitely generated group, for it has a finite generating set, for example $S = \{1\}$. The generated group $\mathbb{Z}$ with respect to the generating set $\{1\}$ is usually designated as $(\mathbb{Z}, \{1\})$. The group $(\mathbb{Q}, +)$ of rational numbers under addition cannot be finitely generated.

Generators provide us with a 'compact' representation of finitely generated groups: i.e. a finite set of elements, which by the application of the group operation, gives us the rest of the group.

2.1.2 Groups represented by graphs

In effect, a representation of a finitely generated group with respect to a chosen set of generators can be realised as a *graph*.

Definition (a Cayley graph). Let (G, S) be a finitely generated group. Then the Cayley graph $\Gamma(G, S)$ of a group G with respect to the choice of S is a directed coloured graph, where vertices are identified with the elements of G and the directed edges of a colour s connect all possible pairs of vertices (x, sx), $x \in G$, $s \in S$.

The vertices of a Cayley graph do not have to be labelled, whereas edges must be coloured if there is more than one generator. The edge corresponding to the multiplication by group element x, which satisfies the condition $x^2 = I$, does not need to be a directed edge.

Example. The Cayley graph for the first example, $(\mathbb{Z}, \{1\})$ is an infinite chain as illustrated in the figure below:

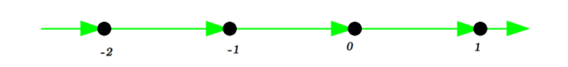

FIGURE 2: The Cayley graph of the group $(\mathbb{Z}, \{1\})$.

Different choices of generators give different Cayley graphs. The same group $\mathbb{Z}$ with generators $\{1, 2\}$ can be depicted as an infinite ladder, as in Figure 3:

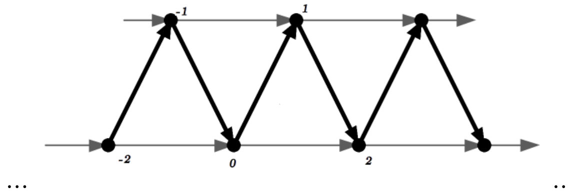

FIGURE 3: The Cayley graph of the group $(\mathbb{Z}, \{1, 2\})$, where bold stands for $\{1\}$.

and ($\mathbb{Z}$, $\{2, 3\}$) in the figure below gives the graph:

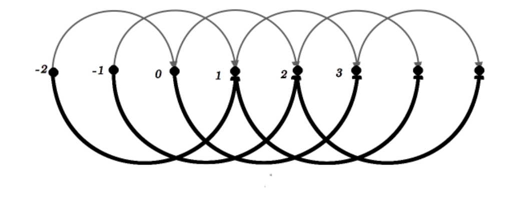

FIGURE 4: The Cayley graph of the group $(\mathbb{Z}, \{2, 3\})$, where bold stands for $\{3\}$.

2.2 Groups and their graphs as metric spaces

The 'geometric' properties of groups mean the properties which can be revealed by thinking of their Cayley graphs as metric spaces (I will explain how). Many of these geometric properties turn out to be independent from the choice of generators for a Cayley graph. For this reason they are considered to be the properties of the groups themselves. At first glance it looks like they depend on the choice of generators, simply because the Cayley graph does. However, it turns out that these properties are the same for different generating sets. Studying these properties makes up a substantial part of geometric group theory. To relate the geometric and algebraic properties of groups means to answer the question:

> If G, G' are quasi-isometric groups, to what extent do G and G' share the same algebraic properties?
>
> In other words, the idea is to 'look at groups through' their Cayley graphs and try to see new (geometric) properties of groups. Then to return to the algebra and check which groups share these properties and under which constraints. This opens up the following opportunities for a group-theorist:

1. Using a presentation of the group G to define a metric on the group and then to exploit the consequent geometry;
2. Defining geometric counterparts to some algebraic properties of groups ('up to quasi-isometry');
3. Classifying groups with these geometric properties.

The obtained results can be considered as a contribution of the groups' geometrisation to the algebra of groups. Now some more precise formulations and examples are to be given.

2.2.1 Word metric

Thinking of groups in terms of metric spaces requires that the relevant space is given by a group G with a particular generating set S and the word metric d_S:$[(G,S),d_S]$. A metric on groups can be introduced by the notion of *word* as defined above.

Definition (a word metric). If $g,h \in G$ then the word metric (with respect to S) $d_S(g,h)$ is the length of a shortest word representing $g^{-1}h$, where $g^{-1}h$ is a word w such that $gw = h$.

The metric space $[(G,S),d_S]$ may not appear at first glance to give us much structure to study as compared to that found in the classical metric spaces, such as the Euclidean or hyperbolic. However it becomes potentially more intriguing when one observes that a discrete valued metric space can be compared to an interesting continuously valued metric space such as the hyperbolic plane. Here is how Gromov expresses this issue about a word metric space:

> This space may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (e.g. topological and infinitesimal) machinery does not run in [the group] Γ. To regain the geometric perspective one has to change one's position and move the observation point far away from Γ. Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \to \infty$ the points in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps and holes and fills our geometer's heart with joy. [Gromov 1993, 1]

This implies that if in a discrete space any pair of distinct points is joined by a geodesic segment (the shortest path), it makes such a metric space comparable to the classical metric spaces in the same way as a sequence of connected points can be similar to a line.[3] This idea is central to geometric group theory, and next I will explain how it was realised in a technical sense.

2.2.2 Cayley graphs as metric spaces

The idea above can be realised through the introduction of a metric on Cayley graphs. By the definition of a Cayley graph, words in a generated group (G, S) correspond to paths in the Cayley graph $\Gamma(G, S)$.

Definition. A *path* between two arbitrary vertices of the graph, x and y, is a sequence of edges between x and y.

The idea is to take each such edge to be of a length equal to 1. Then the length of the path is equal to the number of unit edges in this path. For example, for the Cayley graph in Figure 5 which has one generator 1, the word/path from 0 to 3 is $1 \cdot 1 \cdot 1$, and its length is 3.

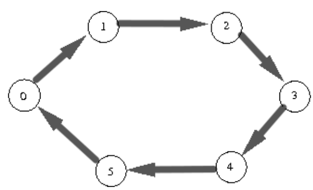

FIGURE 5: The Cayley graph of the group $\mathbb{Z}_6$.

Definition (a path metric). Let $\Gamma(G, S)$ be the Cayley graph for a generated group (G, S). For any pair of vertices x, y of $\Gamma(G, S)$ the *path metric (word metric)* $d_{\Gamma,S}(x, y)$ on the Cayley graph $\Gamma(G, S)$ can be defined as the length of (one of) the shortest paths (*geodesic segment*s) connecting x and y.

The set of vertices in a Cayley graph with this path metric is now a metric space, and the group metric space $((G, S), d_S)$ is isometric to the Cayley graph metric space $(\Gamma(G, S), d_{\Gamma,S})$ by definition.

There can be more than one path (word) that is the shortest in a group. For example, in the Cayley graph of the group $C_3 \times C_2$ in the figure below there are two equally short paths between the vertices labelled 0 and 1: i.e. 0,4,1 and 0,3,1.

3. A geodesic is locally the shortest path between points in the space.

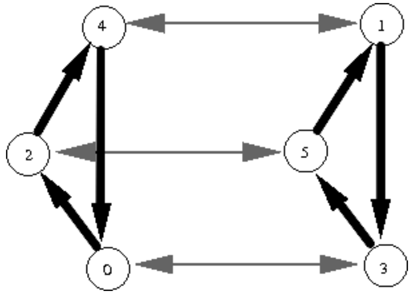

FIGURE 6: The Cayley graph of the group $C_3 \times C_2$.

2.3 Quasi-isometry

The notion of quasi-isometry is central to geometric group theory. It serves to formalise the idea of comparing metric spaces as expressed above by Gromov. Let me use also Bridson's words to better explain its role:

> ... [O]ne needs a language that will lend precision to observations such as the following: if one places a dot at each integer point along a line in the Euclidean plane, then the line and the set of dots become indistinguishable when viewed from afar, whereas the line and the plane remain visibly distinct. One makes this observation precise by saying that the set of dots is quasi-isometric to the line whereas the line is not quasi-isometric to the plane. [Bridson 1999, 138]

2.3.1 Definition and some examples

As Bridson points out, the way to compare metric spaces is through the notion of *quasi-isometry*. It is based on *isometry*, which is a distance-preserving map between metric spaces. Isometry is similar to *congruence* in geometry, in that it expresses the idea 'is the same as' in a given category. To distinguish it further, quasi-isometry is a *weaker* equivalence relation between metric spaces: it is supposed to grasp the idea 'is similar to'.

Definition (a quasi-isometry). For metric spaces (M_1, d_1) and (M_2, d_2) a function $f{:}M_1 \to M_2$ (not necessarily continuous) is called a *quasi-isometry* if there exist constants $A \geq 1$ and $B \geq 0$ such that:

$$\frac{1}{A} d_1(x,y) - B \leq d_2(f(x), f(y)) \leq A d_1(x,y) + B$$

for all x, $y \in M_1$ and a constant $C \geq 0$ such that to every u in M_2 there exists x in M_1 with $d_2(u, f(x)) \leq C$. The spaces M_1 and M_2 are called quasi-isometric if there exists a quasi-isometry $f{:}M_1 \to M_2$. This means that the distance between any two points in M_2 is bounded above and below by linear functions of the distance between the images of these vertices in M_1. The notion of quasi-isometry generalises the familiar definition of isometry corresponding to the case $A = 1$ and $B = 0$.

Examples:

1. Any non-empty bounded space is quasi-isometric to a point: $f : X_1 \to \{x\}$ is a quasi-isometry.
2. $f : \mathbb{R} \times [0,1] \to \mathbb{R}$: projection to the first coordinate is a quasi-isometry, where d is the usual Euclidean metric.
3. $f : (\mathbb{Z}, d) \to (\mathbb{R}, d)$ is a quasi-isometry ($f(x) = x$ for any $x \in \mathbb{Z}$), where d is the usual Euclidean metric (see the figure below). Indeed, $d_2(f(x), f(y)) = d_1(x,y)$. Take $A = 1$ and $B = 0$.

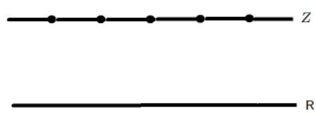

FIGURE 7: $\mathbb{Z}$ is quasi-isomorphic to $\mathbb{R}$.

4. $f : \mathbb{Z}^2 \to E^2$ is a quasi-isometry with respect to the Euclidean metric, as in the figure below:

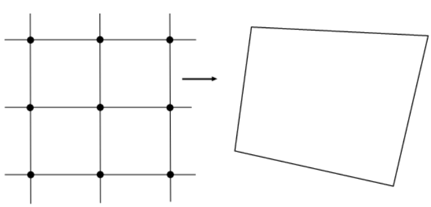

FIGURE 8: $\mathbb{Z}^2$ is quasi-isometric to E^2.

5. The Cayley graph of $(\mathbb{Z}, \{1,2\})$ as well as the Cayley graph of $(\mathbb{Z}, \{2,3\})$ with respect to word metric are quasi-isometric to $\mathbb{R}$ (see Figures 3 and 4).

Non-examples:

1. The empty set is quasi-isometric only to itself. Let us take B to be the empty set, and A to be a non-empty set with some arbitrary metric d. There is no function $f : A \to B$ if B is empty and A is non-empty.
2. Boundedness is a quasi-isometry invariant. Thus for example, $\mathbb{R}$ is not quasi-isometric to $[0, 1]$.

Quasi-isometry on groups The next proposition states that for a given group all its Cayley graphs are quasi-isometric:

Proposition. Take a group G, with finite subsets S and T, such that both S and T generate G. Then the Cayley graph $\Gamma(G, S)$ is *quasi-isometric* to $\Gamma(G, T)$. Indeed, $d_T(x, y) \leq k d_S(x, y)$, where $k = max_i d_T(s_i, I)$, that is k is the greatest length of the generators in the set S measured in the metric d_T, and *vice versa.*

Now it is possible to define quasi-isometry between two finitely generated groups.

Definition (a quasi-isometric group). Two finitely generated groups G and G' are quasi-isometric, if there is a Cayley graph $\Gamma(G, S)$ of the group G with respect to a generating set S, such that $\Gamma(G, S)$ is quasi-isometric to a Cayley graph $\Gamma(G^{'}, T)$ of the group $G^{'}$ with respect to a generating set T.

In other words, two finitely generated groups are quasi-isometric if and only if for some choices of generators in groups G and G', their Cayley graphs are quasi-isometric. The choice of generators is however unimportant because of the previous proposition.

2.3.2 Quasi-isometry invariants: hyperbolicity

It turns out that there is a substantial number of properties in finitely generated groups that are independent from the choice of generators and that are called 'quasi-isometry invariants' or 'geometric properties of groups'. By having a number of geometric properties of groups, one can relate them to the algebraic properties and approach algebraic problems through these geometric properties. One amongst many of these geometric properties is *hyperbolicity.*

The concept of hyperbolic groups was introduced by [Gromov 1987], which has since become very influential and given rise to an extensive research programme. Before this, hyperbolicity was considered only on surfaces and other differentiable manifolds with a metric. Gromov's innovation was to extrapolate it in a more general context, defining it also on discrete objects such as graphs or groups [Gromov 1987]. The idea is again to embed a Cayley graph of a group *quasi-isometrically* into the hyperbolic space $\mathbb{H}^n$ and apply hyperbolic geometry to study this group.[4] This involves the notion of a *geodesic triangle.*

Definition. A geodesic triangle is a figure consisting of three different points together with the pairwise-connecting geodesic segments.

The points are known as the vertices, while the geodesic segments are known as the sides of the triangle. A geodesic triangle can be considered in any space in which geodesics exist. For example, a geodesic triangle in a Cayley graph with the associated word metric consists of three distinct arbitrary vertices x, y, z connected by three geodesic segments (the sides of the triangle), from x to y, y to z and z to x respectively.

Example. In Figure 6, a simple geodesic triangle is formed by the vertices 0,2 and 4. However, the vertices 0,1,2 form two geodesic triangles with the sides 0,2;0,4,1;2,4,1 and 0,2;0,3,1;2,4,1 correspondingly.

Obviously a geodesic triangle (as in the last example) does not have to be shaped as a Euclidean triangle. All what is needed is that three vertices are connected by a geodesic path.

Thin (hyperbolic) triangles and negatively curved groups

Given the notion of a geodesic triangle, one can introduce the notion of *negative curvature* or *hyperbolicity* on Cayley graphs.

The common definition of hyperbolicity is based on the key property of hyperbolic spaces that the sum of a triangle's angles is less than π. In a discrete case as in the case of a graph, there are no angles, yet there is the word metric. The following notion of a *δ-thin triangle* allows us to define hyperbolicity for this metric in a more general way without appealing to the notion of angle. I will now give two definitions of a δ-thin triangle and show how it applies to Cayley graphs and their groups as hyperbolic spaces.

Definition 1 (a δ-thin triangle). Let $\delta \geq 0$. A geodesic triangle in a metric space is said to be δ-thin if each of its sides is contained in the δ-neighbourhood of the union of the other two sides, as demonstrated in Figure 9 below [Gromov 1987, 120].

With the definition of a δ-thin geodesic triangle, we can define a δ-hyperbolic space.

Definition (a hyperbolic space). A geodesic space X is called *δ-hyperbolic* or *negatively curved* if there is a constant δ such that every triangle in X is δ-thin.

The next definition gives a different perspective on thin triangles, which is more clearly related to the idea of quasi-isometry.[5]

4. For details see [Gromov 1987] and [Bridson 1999].
5. For more details see [Bridson 1999, 409].

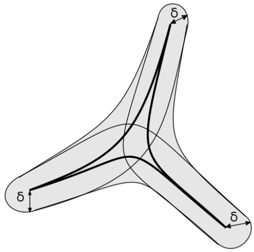

FIGURE 9: A thin hyperbolic triangle.

Definition 2 (a δ-thin triangle). Let $\triangle$ be a geodesic triangle in a metric space X, and $f_\triangle : \triangle \to T_\triangle$ be an isometry from the sides of the triangle $\triangle$ to a tripod $T_\triangle$. Namely, f sends the vertices of the triangle to the vertices of the tripod. For some $\delta \geq 0$, triangle $\triangle$ is δ-thin if $f_\triangle(p) = f_\triangle(q) \Rightarrow d(p,q) \leq \delta$.

The idea is the following. Take any triangle; by pinching the sides of a triangle and making it thinner, one eventually arrives at a tripod as it is visualised in Figure 10:

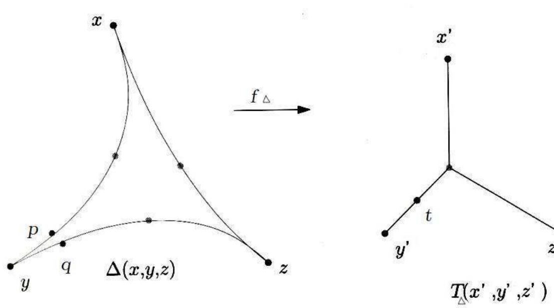

FIGURE 10: Mapping a hyperbolic triangle to a tripod.

Example. In Figure 5, the triangle 1,2,3 forms a tripod. Any of its sides belongs to the union of its other sides. For instance, side 1,2 belongs to the union of the sides 2,3 and 1,2,3. More generally a tripod is a 0-thin geodesic triangle. However, the Cayley graph is hyperbolic because it is bounded (see Example 1 of hyperbolic groups below).

The crucial observation made about the hyperbolicity of groups is that it is an invariant with respect to the choice of generators. Therefore it can be considered as a property of the group as itself and not only a generated group [Gromov 1987, 75–263]. This follows from the next two facts:

1. For any generating sets S and T of a group G, the Cayley graph metric spaces $(\Gamma(G,S),d_S)$ and $(\Gamma(G,T),d_T)$ are quasi-isometric.[6] This fact follows from the proposition in 2.3.1.
2. If two geodesic spaces are quasi-isometric, one is hyperbolic if and only if the other is [Howie 1999, Lemma 3, 10].

It follows from the second definition that those Cayley graphs which are *trees* (along with other graphs satisfying *δ-hyperbolicity)* can be considered as hyperbolic spaces.[7] It is also often said that Gromov-hyperbolic spaces are the ones that exhibit "tree-like behavior".[8]

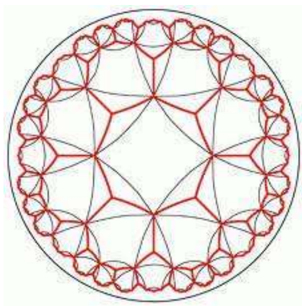

FIGURE 11: A tree on the Poincaré disc.

Examples of hyperbolic groups ([Howie 1999, 11]):

1. Every finite group is hyperbolic, because its Cayley graphs are all bounded. Indeed, for any generating set, the set of vertices in any of its Cayley graphs is finite, as is the distance between any of the vertices. Take δ to be equal to the longest path between all possible pairs of the vertices, then any distance in the graph is no greater than this δ. In particular, the distance between a point on one side of a geodesic triangle and any point on the two other sides is no greater than δ. Therefore this point is in the δ-neighbourhood of the union of the two other sides.
2. Every finitely generated free group is hyperbolic, because its Cayley graphs are trees.[9]

6. See for example [Howie 1999, Example 5, 5].
7. Any connected graph without cycles is a tree, e.g. tripod is a tree.
8. See [Kapovich & Benakli 2002, 31].
9. A group G is called free if it has a subset S such that any element of G can be generated by a unique finite sequence of elements from S and their inverses (disregarding trivial variations such as $sg_1^{-1} = sg_2^{-1}g_2g_1^{-1}$).

3. *A non-example:* Group $\mathbb{Z}^2$ is quasi-isometric to E^2, and hence is not hyperbolic. The fact that $\mathbb{Z}^2$ is not hyperbolic shows that not every finitely generated group is hyperbolic.

Some of the 'geometric properties of groups' independent of the choice of generators are related to important combinatorial problems, such as *finite presentability* and *solvable word problem.*

Finite presentability *Definition. A finitely presented group* is a group with a finite number of generators and relations.

The question is which groups are finitely presentable. It turns out that groups which are quasi-isometric to a finitely presented group are also finitely presented:

Proposition 1. If two groups G and G' are quasi-isometric, then G' is finitely presented if and only if G is finitely presented.[10]

Solvable word problem Suppose a group G is finitely presented. A word in the generators and their inverses represents some element of the group. Is there an algorithm to decide if this is the identity element? Or that two arbitrary words of the group are equivalent? If so, then the group is said to have a solvable word problem. The *uniform* word problem (for the class of all finitely presented groups with solvable word problem) is unsolvable as demonstrated by [Dehn 1911], [Novikov 1955], and [Boone, Cannonito, & Lyndon 1973]. However, the asymptotic approach allows us to formulate some more quantitative statements about the solvability of the word problem in the classes of quasi-isometric groups:

Proposition 2. For two quasi-isometric groups, if one has a solvable word problem, then for the other it is also solvable.

Propositions 1 and 2 connect geometric properties with algebraic properties. Moreover, it was shown that hyperbolicity can give a new perspective on these combinatorial problems. For example, the next two results due to [Gromov 1987] establish important properties of hyperbolic groups related to these problems:

1. Every hyperbolic group has a finite presentation [Howie 1999, Theorem 2, 11–12].
2. Every hyperbolic group has a solvable word problem [Howie 1999, 17, Lemma 6].

Research into this question is still of major importance in combinatorial group theory. Thus this result obtained by the geometric approach about hyperbolic groups is a valuable contribution to the study of groups.

10. See more in [Bridson 1999, 143–144].

3 Discussion

To analyse the case study I will use Manders' model, introduced in the previous section, which sees mathematical activities as the coordination of strategic responses and indifferences to given information. Manders' analysis makes explicit the advantages of Descartes' algebraisation. I will show that it is also effective in my case study.

3.1 Combinatorial vs. geometric representation

To analyse the epistemic role of Cayley graphs in the geometrisation of groups, let me compare them to the ways of representing groups that are traditionally used in combinatorial group theory.

Combinatorial group theory mostly uses symbolic notation. Group elements can be expressed by symbols. For example, the elements of the dihedral group of symmetries in an equilateral triangle D_3 can be written as:

$$I,\ r, r^2, f,\ fr,\ fr^2.$$

For various concrete instances of groups such as D_3, it is possible to have a geometric model of this group. The group elements are three rotations and three flips (in respect to the three heights of the triangle). The following figure demonstrates the rotation r by $\frac{2}{3}\pi$ and one of the flips f, in respect to one of the triangle's heights on a model of an equilateral triangle:

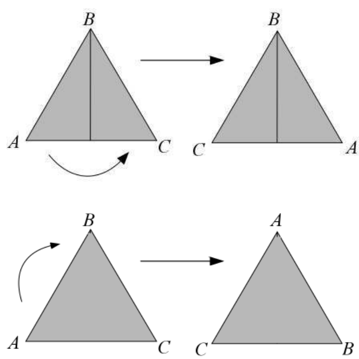

FIGURE 12: The flip f and the rotation r.

All these 'geometric' groups have the same structure as permutation groups, where permutations can also be imagined as transformations in a space (or a plane).

For instance, if we take the equilateral triangle ABC and apply a flip, then this transformation can be presented as a permutation of the set of vertices of ABC: A is switching with C and B is staying the same. Then the rotation r would be C moving to A, B to C and A to B. The flip f followed by rotation $r \bullet f$ can be written by using the following notation:

$$\begin{pmatrix} C & B & A \\ A & C & B \end{pmatrix}\begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

Group elements can be also represented by a multiplication table:

$*$	I	r	r^2	f	fr	fr^2
I	I	r	r^2	f	fr	fr^2
r	r	r^2	I	rf	rfr	rfr^2
r^2	r^2	I	r	r^2f	r^2fr	r^2fr^2
f	f	fr	fr^2	I	r	fr
fr	fr	fr^2	f	frf	$frfr$	$frfr^2$
fr^2	fr^2	f	fr	fr^2f	fr^2fr	fr^2fr^2

TABLE 1: A multiplication table for D_3.

In the latter two cases the algebra represents the concept that was articulated in a geometric way without geometrical allusions: merely in terms of some abstract As and Bs. We can think of group D_3 more abstractly as a set with group axioms, no longer thinking about group elements as continuous motion or as permutations. This is where *indifference* to the geometric aspects is especially effective. The combinatorial representations and axiomatic expression of groups are detached from the concrete nature of the group: one can talk about a group and its properties without specifying what exactly the group operation is. What is important is that this operation satisfies group axioms. This abstract character of algebra allows one to describe the structural aspects of objects from different origins in a flexible way. *Indifference* to geometry provides a broad applicability of group theory to other object-specific fields (e.g. crystallography).

Presented groups are the closest algebraic counterparts to Cayley graphs. Group presentation includes generators and group relations (words equal the identity or *empty words*). Here is the group presentation of D_3: $< r, f \mid r^3, f^2, rfrf >$. To give a more general example: a cyclic group of order n can be presented as $\langle a \mid a^n = e \rangle$. This type of combinatorial representation is quite informative and at the same time, synoptic. It is used both in the combinatorial and in the geometric approach. However, as demonstrated in the previous chapter, for a detailed consideration of group geometry Cayley graph diagrams are essential.

What makes them so effective? For example, the mathematical difference between symbolic notation of a generated group and its Cayley graph is often said to be insignificant, but the cognitive difference between the two cognitive representations is significant. Let us consider how a Cayley graph diagram can be useful in the same example of group D_3 (see Figure 13). Recall that $f^2 = I$. This is immediately visible in the diagram: the two-arrowed edges reflect the fact that $f = f^{-1}$. Similarly, one can easily see that $rf \neq fr$, but it is not the case in Figure 6 and therefore the latter represents an abelian group whereas the former represents a non-abelian group (in an abelian group G, $ab = ba$, $\forall$ $a, b \in G$).

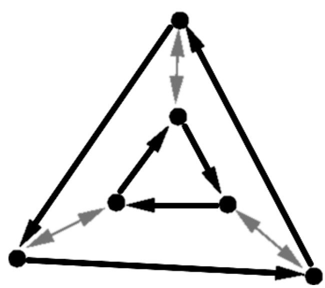

FIGURE 13: The Cayley graph of the group D_3.

Multiplication tables also reflect the abelian property: an abelian group multiplication table is symmetric with respect to the main diagonal. But multiplication tables show only simple relations like $f^2 = I$, whereas in a Cayley graph they are all visible as loops in the diagram. One can check in the diagram if the two paths corresponding to given words end in the same vertex. This would mean that these words are equivalent.

Comparing generated groups with Cayley graphs, geometric group theorists would usually say that the two are 'practically the same objects'. Indeed, what is mathematically significant here is the *response* to the *generating* aspect of the group (the generators). It is essential that a Cayley graph gives *different* information from the information given by the algebraic notation of the generated group: e.g. by colours, edges, shapes, the structure of the group at a glance, and importantly, connectedness. Groups do not have these properties. It is the diagram that suggests our *response* to these properties.

3.2 The *geometric* response to artefacts

As shown in section 2.2.1, groups can be seen as metric spaces equipped with the *word metric*. This is a discrete-valued metric, so it does not give enough structure to compare a word metric space to the classical metric spaces by their geometric properties. However, the practice with particular types of selective responses makes it possible to place groups in the same research-object category as classical metric spaces. This means that groups can be studied by using the same classical methods. The main elements of this practice are (i) the *indifference* to the discrete structure of the group metric space, and (ii) the *response* to the perceptual similarity of particular Cayley graphs with these metric spaces. These are new responses, unavailable to the combinatorial approach. They are implemented by reading the graphs as geometric objects in a space; which in turn leads to new advances—the concepts of quasi-isometry, hyperbolicity of groups and other geometric properties of groups.

The *applied* responses are central to this approach. They are *modified* and *adapted* to the original tasks. In the *geometric* response to Cayley graph diagrams, we perceive them as objects embedded in a space and having geometric elements (e.g. the edges of a graph are thought of as the measurable sides of triangles). This response is applied from geometry, as for example in Euclidean geometry we respond to the intersections of lines, figures and their relations in the diagrams. The response is *modified*: e.g. a geodesic triangle does not have to be shaped like a Euclidean triangle. In other words, some of the Euclidean diagrammatic appearance is *neglected*, whereas the more abstract properties of triangularity (three connected vertices) are *highlighted*. Also for hyperbolic Cayley graphs our response is modified to be non-Euclidean: we see the graph as a geometric object on a saddle-shaped surface or the Poincaré disc (as in Figure 11). Despite the fact that the graphs are visualised as being on a flat surface, we apply *indifference* to this property of the visualisation.

To summarise, practice with various representations helps us to see the different aspects of mathematical objects. Diagrams have some advantages over symbolic notations in terms of representing the structure of objects at a glance, in colours and connected shapes that resemble geometric figures. The examples above demonstrate that the geometric response to the diagrams of graphs makes it clear how we can naturally integrate additional geometric machinery (geodesics, triangles, metric spaces, hyperbolicity). It is easier to think of the geometric and topological properties of an object such as a graph equipped with a diagram, than of an algebraic one equipped with algebraic symbolism. As a result, a particular response to diagrammatic representations facilitates the ongoing application and development of concepts.

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